

# CONSTANT RANK-DISTANCE SETS OF HERMITIAN MATRICES AND PARTIAL SPREADS IN HERMITIAN POLAR SPACES

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**ABSTRACT.** In this paper we investigate partial spreads of  $H(2n-1, q^2)$  through the related notion of partial spread sets of hermitian matrices, and the more general notion of constant rank-distance sets. We prove a tight upper bound on the maximum size of a linear constant rank-distance set of hermitian matrices over finite fields, and as a consequence prove the maximality of extensions of symplectic semifield spreads as partial spreads of  $H(2n-1, q^2)$ . We construct large constant rank-distance sets for all even ranks, and construct maximal partial spreads of  $H(3, q^2)$  for a range of sizes.

## 1. INTRODUCTION

A *partial spread* of a projective or polar space  $P$  is a set  $S$  of pairwise disjoint  $(t-1)$ -dimensional subspaces of  $P$ . A partial spread is called a *spread* if the elements of  $S$  cover  $P$ .

A *partial spread set*  $U$  is a set of  $n \times n$  matrices such that

$$\begin{aligned}\text{rank}(A - B) &= n \text{ for all } A, B \in U, A \neq B; \\ \text{rank}(A) &= n \text{ for all } A \in U, A \neq 0.\end{aligned}$$

It is well known that a partial spread set in  $M_n(F)$  defines a partial spread in the projective space  $PG(2n-1, F)$ ; a partial spread set in the space of hermitian matrices  $H_n(K)$  defines a partial spread in the hermitian polar space  $H(2n-1, K)$ ; and a partial spread set in the space of symmetric matrices  $S_n(F)$  defines a partial spread in the symplectic polar space  $W(2n-1, F)$ . We will recall these connections in Section 2.

In this paper we investigate partial spread sets as a special case of the following more general definition:

A *constant rank-distance  $k$  set* is a set of  $n \times n$  matrices  $U$  such that

$$\begin{aligned}(i) \quad &\text{rank}(A - B) = k \text{ for all } A, B \in U, A \neq B; \\ (ii) \quad &\text{rank}(A) = k \text{ for all } A \in U, A \neq 0.\end{aligned}$$

This name follows the definition of *rank-distance* in [9]. Clearly a constant rank-distance  $n$  set is a partial spread set. Note that a set satisfying only property (i) implies the existence of a constant rank-distance  $k$  set of the same size. For

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suppose  $U'$  is such a set, and choose some  $A \in U'$ . Then it is easily verified that  $U := \{A - B : B \in U'\}$  is a constant rank-distance  $k$  set.

A constant rank-distance  $k$  set (resp. partial spread set) is said to be *maximal* if it is not strictly contained in a larger constant rank-distance  $k$  set (resp. partial spread set).

If a constant rank-distance  $k$  set (resp. partial spread set) is closed under  $F'$ -linear combinations for some field  $F'$ , we refer to it as an  $F'$ -linear constant rank-distance  $k$  set (resp. partial spread set). We will often omit the specification of a particular field.

In Section 3 we will prove a new bound for linear constant rank-distance sets of hermitian matrices over finite fields, using properties of characters defined on function spaces. In Section 4, we will show how this theorem provides new results on maximal partial spreads of  $H(2n-1, q^2)$ . In Section 4, we will construct maximal partial spreads of  $H(3, q^2)$  of a range of different sizes. Finally in Section 5, we will construct some constant rank-distance sets which are larger than the largest possible linear constant rank-distance sets.

## 2. PARTIAL SPREADS AND SUBSPACE CODES

We recall now the connection between partial spreads and partial spread sets ([6], p. 220), and the definition of a subspace code. Given an  $n \times n$  matrix  $A$ , we can define an  $(n-1)$ -dimensional subspace  $S_A$  of  $PG(2n-1, q)$  as follows:

$$S_A := \left\langle \begin{bmatrix} u \\ Au \end{bmatrix} : u \in \mathbb{F}_q^n \right\rangle$$

Given any two matrices  $A$  and  $B$ , it is easy to see that

$$\dim(S_A \cap S_B) = n - \text{rank}(A - B) - 1,$$

and  $S_A \cap S_B = \emptyset$  if and only if  $\text{rank}(A - B) = n$ . We define also the  $(n-1)$ -dimensional subspace  $S_\infty$  of  $PG(2n-1, q)$  by

$$S_\infty := \left\langle \begin{bmatrix} 0 \\ u \end{bmatrix} : u \in \mathbb{F}_q^n \right\rangle.$$

Again it is clear that  $S_A \cap S_\infty = \emptyset$  for all  $A \in M_n(\mathbb{F}_q)$ .

Hence if  $U$  is a partial spread set in  $M_n(\mathbb{F}_q)$ , then the set

$$D_U := \{S_A : A \in U\} \cup \{S_\infty\}$$

is a partial spread of  $\mathbb{F}_q^{2n}$ , with  $|D_U| = |U| + 1$ .

Every spread  $D$  of  $PG(2n-1, q)$  defines a *translation plane*, via the André-Bruck-Bose construction. If the dual of this plane is also a translation plane, it is called a *semifield plane*. The algebraic structure which coordinatizes this plane is then a *semifield*. This occurs if and only if the spread set of  $D$  is linear over some field. For this reason, a linear spread set is also called a *semifield spread set*. See [13].

A *hermitian polar space*  $H(t-1, q^2)$  is the geometry of subspaces of  $PG(t-1, q^2)$  which are totally isotropic with respect to some non-degenerate hermitian form on  $\mathbb{F}_{q^2}^t$ . It is well known that the maximum dimension of a subspace contained in  $H(t-1, q^2)$  is  $\lfloor \frac{t-1}{2} \rfloor$ .

The projective geometry  $PG(t-1, q^2)$  contains  $PG(t-1, q)$  as a subgeometry. A *symplectic polar space*  $W(t-1, q)$  is the geometry of subspaces in  $PG(t-1, q)$  which are isotropic with respect to some non-degenerate symplectic form on  $\mathbb{F}_q^t$ . It is necessary that  $t$  be even.

Consider now the space  $H(2n-1, q^2)$ . We take the defining nondegenerate hermitian form to be

$$u \mapsto \overline{u}^T \begin{bmatrix} 0_n & aI_n \\ -aI_n & 0_n \end{bmatrix} u,$$

where  $a \in \mathbb{F}_{q^2}$  is such that  $\overline{a} = -a$ . Then given a matrix  $A \in M_n(\mathbb{F}_{q^2})$ , the space  $S_A$  is contained in  $H(2n-1, q^2)$  if and only if it is hermitian: for

$$\begin{bmatrix} \overline{u}^T & \overline{u}^T \overline{A}^T \end{bmatrix} \begin{bmatrix} 0_n & aI_n \\ -aI_n & 0_n \end{bmatrix} \begin{bmatrix} u \\ Au \end{bmatrix} = 0$$

for all  $u \in \mathbb{F}_{q^2}^n$  if and only if

$$a\overline{u}^T (A - \overline{A}^T) u = 0$$

for all  $u \in \mathbb{F}_{q^2}^n$ , if and only if  $A = \overline{A}^T$ , as claimed. Note that  $A \mapsto \overline{A}$  denotes the Frobenius automorphism  $x \mapsto x^q$  applied entrywise.

Hence a partial spread set  $U$  in  $H_n(\mathbb{F}_{q^2})$  leads to a partial spread  $D_U$  in  $H(2n-1, q^2)$ , with  $|D_U| = |U| + 1$ . Conversely, it is well known that every partial spread  $D$  in  $H(2n-1, q^2)$  is isomorphic to  $D_U$  for some partial spread set  $U$  in  $H_n(\mathbb{F}_{q^2})$ , [6].

Clearly the intersection of  $H(2n-1, q^2)$  with  $PG(2n-1, q)$  is a symplectic polar space  $W(2n-1, q)$ , with defining symplectic form

$$u \mapsto u^T \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} u.$$

A partial spread in  $W(2n-1, q)$  then leads to a partial spread set in  $S_n(\mathbb{F}_q)$ , which is of course an  $\mathbb{F}_q$ -subspace of  $H_n(\mathbb{F}_{q^2})$ . In fact it is also a partial spread set in  $H_n(\mathbb{F}_{q^2})$ , as the rank of a matrix does not change over an extension field. We refer to the associated partial spread of  $H(2n-1, q^2)$ , as the *extension* of the partial spread of  $W(2n-1, q)$ .

Spreads exist in  $W(2n-1, q)$  for all  $q, n$ , and have size  $q^n + 1$ . In fact *linear* spreads exist in  $W(2n-1, q)$  for all  $q, n$ , as Kantor [11] showed that a linear spread in  $PG(2n-1, q)$  is symplectic if and only if the semifield it defines is Knuth-equivalent to a *commutative semifield*. Such a spread is called a *symplectic semifield spread*, see [13].

Much study has been dedicated to the maximum size of a partial spread in  $H(2n-1, q^2)$ , or equivalently the maximum size of a partial spread set in  $H_n(\mathbb{F}_{q^2})$ . Particular attention has been paid to the case  $H(3, q^2)$ , or equivalently  $H_2(\mathbb{F}_{q^2})$ . See [4] for an overview of the known results.

A *subspace code* is a set of subspaces of  $PG(N, F)$  together with the distance function  $d(S, T) = \dim(S+T) - \dim(S \cap T)$ . A subspace code  $C$  such that  $\dim(S) = t$  for all  $S \in C$  is called a *constant dimension code*. Subspace codes and constant dimension codes are of interest in network coding, see for example [12] for a survey, and the connection with sets of matrices with particular rank properties (“rank metric codes”) has been studied in for example [20]. Note that subspace codes are

normally considered as subspaces of the vector space  $F^{N+1}$ . However the constructions are equivalent.

Given a constant rank-distance  $k$  set  $U$  of  $n \times n$  matrices, define the subset  $D'_U = \{S_A : A \in U\} \subseteq PG(2n-1, F)$ , where  $S_A$  is defined as above. Then  $D'_U$  is a constant dimension  $n$  code. Moreover,  $d(S_A, S_B) = 2k$  for all  $A, B \in U$ . Hence  $D'_U$  is also a *constant distance code*, and  $|D'_U| = |U|$ . Note that if  $k < n$  we do not include the space  $S_\infty$ , as  $d(S_A, S_\infty) = 2n \neq 2k$ .

When  $k = n$  we can include  $S_\infty$ , and  $D_U = D'_U \cup \{S_\infty\}$  is a partial spread, or equivalently a constant dimension  $n$ , constant distance  $2n$  code. If  $C$  is also a spread, it is more commonly referred to as a *spread code*. See for example [15] for more on spread codes. While subspace codes are normally studied in  $PG(N, q)$ , in this work we will focus on those in  $H(2n-1, q^2)$ .

We summarise this discussion in the following lemmas for future reference.

**Lemma 1.** *There exists a partial spread set in  $H_n(\mathbb{F}_{q^2})$  of size  $N$  if and only if there exists a partial spread in  $H(2n-1, q^2)$  of size  $N+1$ .*

**Lemma 2.** *If there exists a constant rank-distance  $k$  set in  $H_n(\mathbb{F}_{q^2})$  of size  $N$ , then there exists a constant dimension  $n$ , constant distance  $2k$  code in  $H(2n-1, q^2)$  of size  $N$ .*

### 3. CHARACTER THEORY AND CONSTANT RANK-DISTANCE SETS

We will prove a new upper bound on linear constant rank-distance sets by considering (hermitian) matrices as elements of the function space  $\mathbb{F}_q^\Omega$  for appropriate choices of a finite set  $\Omega$ . Throughout the rest of this paper we will assume  $q = p^e$  for some prime  $p$  and positive integer  $e$ .

For any  $f \in \mathbb{F}_q^\Omega$ , i.e. any map  $f : \Omega \rightarrow \mathbb{F}_q$ , and any  $a \in \mathbb{F}_q$ , we define the number

$$N_f(a) := \#\{\omega \in \Omega \mid f(\omega) = a\}.$$

We denote  $V = \mathbb{F}_q^n$ ,  $V' = \mathbb{F}_q^m$  and  $W = \mathbb{F}_{q^2}^n$ . We will represent each of these spaces as column vectors; e.g.  $V$  will be the vector space of  $n \times 1$  matrices with entries in  $\mathbb{F}_q$ .

We recall now the well known one-to-one correspondence between  $m \times n$  matrices and *bilinear forms* on  $V' \times V$ . Let  $M_{m \times n}(\mathbb{F}_q)$  denote the set of  $m \times n$  matrices with entries in  $\mathbb{F}_q$ . For each matrix  $A$  we define a bilinear form (which by abuse of notation we will also denote by  $A$ ), by

$$\begin{aligned} A(v', v) : V' \times V &\rightarrow \mathbb{F}_q \\ &: (v', v) \mapsto v'^T A v. \end{aligned}$$

We denote the space of bilinear forms on  $V' \times V$  by  $\mathcal{B}_{m,n}$ . Then we have

$$M_{m \times n}(\mathbb{F}_q) \simeq \mathcal{B}_{m,n} \leq \mathbb{F}_q^{V' \times V}.$$

We define the *rank* of a bilinear form to be the rank of the associated matrix. The following lemma is well known, a proof can be found in [19], Lemma 3.3.

**Lemma 3.** *For any bilinear form  $A$ , with  $\text{rank}(A) = r$ ,*

$$N_A(a) = \begin{cases} q^{m+n-r-1}(q^r + q - 1) & \text{if } a = 0 \\ q^{m+n-r-1}(q^r - 1) & \text{otherwise} \end{cases}$$

Now let  $H_n(\mathbb{F}_{q^2})$  denote the set of  $n \times n$  hermitian matrices with entries in  $\mathbb{F}_{q^2}$ , i.e.

$$H_n(\mathbb{F}_{q^2}) = \{h \in M_n(\mathbb{F}_{q^2}) \mid h = \overline{h}^T\}.$$

Recall that  $H_n(\mathbb{F}_{q^2})$  is a vector space over  $\mathbb{F}_q$ , but *not* over  $\mathbb{F}_{q^2}$ . To each hermitian matrix  $h$  we associate a *quadratic hermitian form* (which by abuse of notation we will also denote by  $h$ ), by

$$\begin{aligned} h : W &\rightarrow \mathbb{F}_q \\ w &\mapsto \overline{w}^T h w. \end{aligned}$$

Note that  $h(w)$  does indeed lie in  $\mathbb{F}_q$  for all  $w \in W$ , as  $\overline{h(w)} = \overline{h(w)}^T = \overline{w^T h^T w} = \overline{w}^T \overline{h}^T w = \overline{w}^T h w = h(w)$ .

We denote the set of quadratic hermitian forms by  $\mathcal{H}_n$ , and we have

$$H_n(\mathbb{F}_{q^2}) \simeq \mathcal{H}_n \leq \mathbb{F}_q^W.$$

Again we define the *rank* of a quadratic hermitian form to be the rank of the associated hermitian matrix. The proof of the following well-known lemma can also be found in [19].

**Lemma 4.** *For any quadratic hermitian form  $h$ ,  $\text{rank}(h) = r$ , we have*

$$N_h(a) = \begin{cases} q^{2n-r-1}(q^r + (-1)^r(q-1)) & \text{if } a = 0 \\ q^{2n-r-1}(q^r - (-1)^r) & \text{otherwise.} \end{cases},$$

where  $r = \text{rank}(h)$ .

We now view  $(\mathbb{F}_q^\Omega, +)$  as a finite group. We will denote the identity element of this group (the zero function, which maps all elements of  $\Omega$  to zero), by  $f_0$ . We can define some linear characters on  $\mathbb{F}_q^\Omega$  as follows. Let  $\epsilon$  be a primitive  $p^{\text{th}}$  root of unity in  $\mathbb{C}$  (where  $q$  is some power of the prime  $p$ ), and let  $\text{tr}$  denote the trace map from  $\mathbb{F}_q$  to  $\mathbb{F}_p$ . For each  $\omega \in \Omega$ , define the function  $\chi_\omega : \mathbb{F}_q^\Omega \rightarrow \mathbb{C}$  by

$$\chi_\omega : f \mapsto \epsilon^{\text{tr}(f(\omega))}.$$

It is clear that  $\chi_\omega$  is a linear character, as  $\chi_\omega(f + g) = \chi_\omega(f) + \chi_\omega(g)$  for all  $f, g \in \mathbb{F}_q^\Omega$ , and  $\chi_\omega(f_0) = \epsilon^0 = 1$ .

Define now a character  $\chi_\Omega$  by

$$\chi_\Omega := \sum_{\omega \in \Omega} \chi_\omega.$$

We will mostly simply write  $\chi$  for  $\chi_\Omega$  when there is no ambiguity. Then by definition it is clear that

$$\chi(f) = \sum_{\omega \in \Omega} \epsilon^{\text{tr}(f(\omega))} = \sum_{a \in \mathbb{F}_q} N_f(a) \epsilon^{\text{tr}(a)}.$$

This leads to the following lemma.

**Lemma 5.** *Suppose  $f$  is a bilinear form of rank  $k$  on  $V \times V'$ , and let  $\chi = \chi_{V \times V'}$ . Then*

$$\chi(f) = q^{m+n-k}.$$

*Proof.* By the above formula, we have  $\chi(f) = \sum_{a \in \mathbb{F}_q} N_f(a) \epsilon^{tr(a)}$ . But  $N_f(0) = q^{m+n-k-1}(q^k + q - 1)$ , and  $N_f(a) = q^{m+n-k-1}(q^k - 1)$  for all non-zero  $a \in \mathbb{F}_q$ , and so

$$\chi(f) = q^{m+n-k-1}(q^k + q - 1) + q^{m+n-k-1}(q^k - 1) \sum_{a \in \mathbb{F}_q^\times} \epsilon^{tr(a)}.$$

But  $\sum_{a \in \mathbb{F}_q} \epsilon^{tr(a)} = 0$ , and hence  $\sum_{a \in \mathbb{F}_q^\times} \epsilon^{tr(a)} = -1$ , giving us

$$\chi(f) = q^{m+n-k-1}(q^k + q - 1) - q^{m+n-k-1}(q^k - 1) = q^{m+n-k},$$

proving the claim.  $\square$

Similarly we have the following lemma.

**Lemma 6.** *Suppose  $f$  is a quadratic hermitian form of rank  $k$  on  $W = \mathbb{F}_q^n$ , and let  $\chi = \chi_W$ . Then*

$$\chi(f) = (-1)^k q^{2n-k}.$$

*Proof.* Again we have  $\chi(f) = \sum_{a \in \mathbb{F}_q} N_f(a) \epsilon^{tr(a)}$ . But  $N_f(0) = q^{2n-k-1}(q^k + (-1)^k(q - 1))$ , and  $N_f(a) = q^{2n-k-1}(q^k - (-1)^k)$  for all non-zero  $a \in \mathbb{F}_q$ , and so

$$\chi(f) = q^{2n-k-1}(q^k + (-1)^k(q - 1)) + q^{2n-k-1}(q^k - (-1)^k) \sum_{a \in \mathbb{F}_q^\times} \epsilon^{tr(a)}.$$

But again  $\sum_{a \in \mathbb{F}_q^\times} \epsilon^{tr(a)} = -1$ , giving us

$$\chi(f) = q^{2n-k-1}(q^k + (-1)^k(q - 1)) - q^{2n-k-1}(q^k - (-1)^k) = (-1)^k q^{2n-k},$$

proving the claim.  $\square$

Recall the definition of the inner product of characters on any group  $G$ :

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.$$

It is well known that this number is a non-negative integer. The restriction of a character on a group  $G$  to a subgroup  $H$  is again a character on  $H$ . We denote the trivial character by  $1_G$ , i.e.  $1_G(g) = 1$  for all  $g \in G$ .

Suppose now we have a subgroup  $U$  of  $(\mathbb{F}_q^\Omega, +)$ . Clearly  $U$  is an  $\mathbb{F}_p$ -subspace of  $\mathbb{F}_q^\Omega$ . We will use the restriction of the previously defined characters on  $(\mathbb{F}_q^\Omega, +)$  to  $U$  to obtain upper bounds on the sizes of certain classes of subgroups.

Now let  $\Omega = V' \times V$ , and suppose  $U \leq \mathcal{B}_{m,n} \leq \mathbb{F}_q^\Omega$ , where “ $\leq$ ” denotes subgroup (or equivalently,  $\mathbb{F}_p$ -subspace). Then we have the following:

**Lemma 7.** *Let  $U$  be an  $\mathbb{F}_p$ -subspace of  $\mathcal{B}_{m,n}$ , with  $|U| = p^d$ . Let  $A_k$  denote the number of elements of  $U$  of rank  $k$ . Then*

$$\sum_{k=0}^n A_k p^{(m+n-k)e-d}$$

*is a non-negative integer.*

*Proof.* Let  $\chi = \chi_{V' \times V}$ . Taking the inner product  $\langle \chi|_U, 1_U \rangle$  gives us

$$\langle \chi|_U, 1_U \rangle = \frac{1}{p^d} \sum_{f \in U} \chi(f).$$

But by Lemma 5,  $\chi(f) = q^{m+n-k} = p^{(m+n-k)e}$ , where  $\text{rank}(f) = k$ , and so

$$\langle \chi|_U, 1_U \rangle = \sum_{k=0}^n A_k p^{(m+n-k)e-d}$$

is a non-negative integer, proving the result.  $\square$

This leads to the following.

**Theorem 1.** *Let  $U$  be a linear constant rank-distance  $k$  subspace of  $M_{m \times n}(\mathbb{F}_q)$ . Then*

$$|U| \leq q^{m+n-k}.$$

*Proof.* Consider  $U$  as an  $\mathbb{F}_p$ -subspace of  $\mathcal{B}_{m,n}$ , and let  $A_i$  be as in Lemma 7. If  $|U| = p^d$ , we have that  $A_0 = 1$ ,  $A_k = p^d - 1$ ,  $A_i = 0$  otherwise. Hence by Lemma 7, the number

$$p^{(m+n-k)e-d} (p^{ke} + p^d - 1)$$

must be a non-negative integer. But  $p^{ke} + p^d - 1$  is a positive integer relatively prime to  $p$ , and so  $p^{(m+n-k)e-d}$  must be an integer, implying  $d \leq (m+n-k)e$ , and  $|U| \leq p^{(m+n-k)e} = q^{m+n-k}$ , as claimed.  $\square$

Note that this character is related to the characters considered by Delsarte in [5] in the following way. Delsarte considered characters on the group  $M_{m \times n}(\mathbb{F}_q)$  of the form

$$P_X : M_{m \times n}(\mathbb{F}_q) \rightarrow \mathbb{C} \\ A \mapsto \epsilon^{\text{tr}(\text{Tr}(XA))},$$

where  $X \in M_{n \times m}(\mathbb{F}_q)$ . Here  $\text{Tr}$  denotes matrix trace, and  $\text{tr}$  denotes field trace. He then defined characters  $P_k$  by

$$P_i := \sum_{\{X | \text{rank}(X)=i\}} P_X.$$

He showed ([5], Theorem A2) that each character  $P_i$  takes the same integer value on elements of the same rank, and calculated these integers. The restriction of our above defined character  $\chi$  on  $\mathbb{F}_q^{V' \times V}$ , is related to the character  $P_1$  as follows: every

rank one matrix  $X$  can be written as  $uv^T$  for some  $u \in V'$ ,  $v \in V$ , and there are precisely  $(q-1)$  pairs  $(u, v)$  such that  $X = uv^T$ . Now

$$\begin{aligned}\phi_X(A) &= \epsilon^{\text{tr}(\text{Tr}(XA))} \\ &= \epsilon^{\text{tr}(\text{Tr}(uv^T A))} \\ &= \epsilon^{\text{tr}(\text{Tr}(v^T Au))} \\ &= \epsilon^{\text{tr}(v^T Au)} \\ &= \chi_{(u,v)}(A).\end{aligned}$$

Hence we have that

$$\chi = (q-1)P_1 - (q^m + q^n - 1)1_M,$$

where  $1_M$  denotes the trivial character. In [7], the character values  $P_1$  were used to prove Theorem 1. This new self-contained proof will now be generalized to hermitian matrices.

We now let  $\Omega = W$ , and consider subgroups  $U \leq \mathcal{H}_n \leq \mathbb{F}_q^\Omega$ .

**Lemma 8.** *Let  $U$  be an  $\mathbb{F}_p$ -subspace of  $\mathcal{H}_n$ , with  $|U| = p^d$ . Let  $A_k$  denote the number of elements of  $U$  of rank  $k$ . Then*

$$\sum_{k=0}^n (-1)^k A_k p^{(2n-k)e-d}$$

*is a non-negative integer.*

*Proof.* Let  $\chi = \chi_W$ . Taking the inner product  $\langle \chi|_U, 1_U \rangle$  gives us

$$\langle \chi|_U, 1_U \rangle = \frac{1}{p^d} \sum_{f \in U} \chi(f).$$

But by Lemma 6,  $\chi(f) = (-1)^k q^{2n-k} = (-1)^k p^{(2n-k)e}$ , where  $\text{rank}(f) = k$ , and so

$$\langle \chi|_U, 1_U \rangle = \sum_{k=0}^n (-1)^k A_k p^{(2n-k)e-d}$$

is a non-negative integer, proving the result.  $\square$

This leads to the following.

**Theorem 2.** *Let  $U$  be a linear constant rank-distance  $k$  set of  $H_n(\mathbb{F}_{q^2})$ . Then*

$$|U| \leq \begin{cases} q^k & \text{if } k \text{ is odd} \\ q^{2n-k} & \text{if } k \text{ is even} \end{cases}$$

*Proof.* Consider  $U$  as a  $\mathbb{F}_p$ -subspace of  $\mathcal{H}_n$ , and let  $A_i$  be as in Lemma 8. If  $|U| = p^d$ , we have that  $A_0 = 1$ ,  $A_k = p^d - 1$ ,  $A_i = 0$  otherwise. Hence by Lemma 8, the number

$$p^{(2n-k)e} + (-1)^k (p^d - 1) p^{(2n-k)e-d} = p^{(2n-k)e-d} (p^{ke} + (-1)^k (p^d - 1))$$

must be a non-negative integer. Hence if  $k$  is odd, we must have  $p^{ke} \geq p^d$ , and hence  $|U| \leq q^k$ , as claimed. If  $k$  is even, then  $p^{ke} + p^d - 1$  is a positive integer relatively prime to  $p$ , and so  $p^{(2n-k)e-d}$  must be an integer, implying  $d \leq (2n-k)e$ , and  $|U| \leq p^{(2n-k)e} = q^{2n-k}$  as claimed.  $\square$



In [8], Theorem 2 was proved for the special case where  $U$  is an  $\mathbb{F}_q$ -subspace. Hence this theorem is a generalisation of that result. It was also shown that this bound is met in all cases. In the next two sections we will consider non-linear constant rank-distance sets which exceed these bounds.

#### 4. MAXIMAL PARTIAL SPREADS OF $H(2n - 1, q^2)$

In this section we apply Theorem 2 to prove new results on the maximality of some partial spreads of  $H(2n - 1, q^2)$ , and construct new maximal partial spreads in  $H(3, q^2)$ .

Thas [21] showed that spreads do not exist in  $H(2n - 1, q^2)$ . Much study has been dedicated to the spectrum of sizes of maximal partial spreads, see for example [4].

As noted in Section 2, spreads in  $W(2n - 1, q)$  lead to partial spreads in  $H(2n - 1, q^2)$  of the same size. Such a spread always exists, and has order  $q^n + 1$ . If  $n$  is odd, this is in fact the largest possible size of a partial spread in  $H(2n - 1, q^2)$ :

**Theorem 3** (Vanhove). *A partial spread in  $H(2n - 1, q^2)$ ,  $n$  odd, has size at most  $q^n + 1$ .*

This was proved by Vanhove using graph-theoretical techniques in [22], and again geometrically in [24]. Neither of these techniques extend to the case where  $n$  is even, and in fact we will see in Remark 2 that partial spreads of size larger than  $q^n + 1$  always exist.

Aguglia, Cossidente and Ebert [1] proved the following (although their terminology is different).

**Theorem 4** (Aguglia-Cossidente-Ebert). *Any extension of a spread in  $W(3, q)$  to a partial spread in  $H(3, q^2)$  is maximal.*

Theorem 2 above now gives the following new result.

**Theorem 5.** *Any extension of a semifield spread in  $W(2n - 1, q)$  to a partial spread in  $H(2n - 1, q^2)$  is maximal.*

*Proof.* The spread set  $U$  of a semifield spread in  $W(2n - 1, q)$  is by definition a linear spread set in  $S_n(\mathbb{F}_q)$ , and  $|U| = q^n$ . Consider  $U$  now as a partial spread set in  $H_n(\mathbb{F}_{q^2})$ . Suppose there exists some  $A \in H_n(\mathbb{F}_{q^2})$  such that  $U \cup A$  is a partial spread set,  $A \notin U$ . Then  $\det(A - B) \neq 0$  for all  $B \in U$ . But then  $\det(\lambda A - B) \neq 0$  for all  $B \in U$ ,  $\lambda \in \mathbb{F}_p^\times$ , and so  $\langle A, U \rangle$  would be a linear partial spread set of size  $p^{nh+1}$ , contradicting Theorem 2. Hence  $U$  is a maximal partial spread set.  $\square$

The question remains open whether the extension of every (non-linear) symplectic spread in  $W(2n - 1, q)$  is a maximal partial spread in  $H(2n - 1, q^2)$  for  $n$  even,  $n > 2$ .

We now turn our attention to the question of the existence of an interval of integers such that, for each integer contained, there exists a maximal partial spread of that size. This question has received attention for the case of  $PG(3, q)$  ([10]),  $W(3, q)$  and  $Q(4, q)$  ([17], [16], [18]). We now construct maximal partial spreads in  $H(3, q^2)$  for a range of sizes. Known results on the spectrum of sizes of maximal partial

spreads in  $H(3, q^2)$  can be found in for example [4], [2]. These partial spreads have received particular attention due to their equivalence with partial ovoids in the elliptic quadric  $Q^-(5, q)$ . Though these new maximal partial spreads are not the largest nor smallest known, the authors know of no other constructions for an interval of sizes in this space.

**Theorem 6.** *There exists a maximal partial spread in  $H(3, q^2)$  of size  $N$  for every integer  $N$  in the interval  $[q^2 + 1, q^2 + q]$ .*

*Proof.* Let  $\delta$  be some integer in  $\{1, \dots, q\}$ . Choose some arbitrary subset  $\Delta$  of  $\mathbb{F}_q$  of size  $\delta$ , containing 0. Then define the set

$$U_\delta = \left\{ \begin{bmatrix} a & a \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & a \\ a & \mu a \end{bmatrix} : a \in \Delta \right\} \cup \left\{ \begin{bmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{bmatrix} : \alpha \in \mathbb{F}_{q^2} - \Delta \right\},$$

where  $\mu$  is chosen such that  $x^2 + y^2 + (\mu - 2)xy = 0$  has no solutions in  $\mathbb{F}_q$ . We claim that  $U_\delta$  is a maximal partial spread set in  $H_2(\mathbb{F}_{q^2})$ , and so  $D_{U_\delta}$  is a maximal partial spread in  $H(3, q^2)$  and  $|D_{U_\delta}| = |U_\delta| + 1 = q^2 + \delta$ .

The fact that this is a partial spread set is easily verified. It remains to show that it is maximal. First note that

$$\{N(y - \alpha) : \alpha \in \mathbb{F}_{q^2}/\Delta\} = \begin{cases} \mathbb{F}_q^\times & \text{if } y \in \Delta \\ \mathbb{F}_q & \text{otherwise} \end{cases}$$

This is because, counting multiplicities, this set has size  $q^2 - \delta \geq q^2 - q$ . Clearly 0 has multiplicity 0 in the first case and 1 in the second. If some non-zero  $\lambda \in \mathbb{F}_q$  were not in this set, it could have size at most  $(q - 2)(q + 1) + 1 = q^2 - q - 1$ , a contradiction.

Now suppose there exists some  $\begin{bmatrix} x & y \\ \bar{y} & z \end{bmatrix}$  which extends  $U_\delta$ . Then  $xz \notin \{N(y - \alpha) : \alpha \in \mathbb{F}_{q^2}/\Delta\}$ . By the previous argument, we must have  $y \in \Delta$  and  $xz = 0$ . Therefore either  $x = 0$  or  $z = 0$ . Suppose first  $z = 0$ . Then

$$\begin{bmatrix} x & y \\ y & 0 \end{bmatrix} - \begin{bmatrix} y & y \\ y & 0 \end{bmatrix}$$

is not invertible, a contradiction. Similarly in the case  $x = 0$  we get a contradiction, proving that such a matrix can't exist, thus proving maximality.

It is clear that  $|U_\delta| = (q^2 - \delta) + (2\delta - 1) = q^2 + \delta - 1$ . □

We will see in the next section that there always exist partial spreads of  $H(2n - 1, q^2)$  of size greater than  $q^n + 1$  for all  $n$  even.

## 5. LARGE CONSTANT RANK-DISTANCE SETS IN $H_n(\mathbb{F}_{q^2})$

Constant rank-distance sets of rank less than  $n$  have received far less attention than partial spread sets, with most of the focus applied to subspaces. See [19] and the references therein. However the following theorem, which follows from [23], Lemma 3.2 (and also follows from the proof of [22] Theorem 4.2), gives an upper bound when  $n$  is *odd*. Note that the result in [23] is much more general, however here we only state the portion relevant to this work, and we restate the result in the language of this paper.

**Theorem 7** (Vanhove). *A constant dimension  $n$ , constant distance  $2k$  code in  $H(2n-1, q^2)$ ,  $k$  odd, has size at most  $q^k + 1$ .*

**Corollary 1.** *A constant rank-distance  $k$  set in  $H_n(\mathbb{F}_{q^2})$  has size at most  $q^k + 1$ .*

Recall that there exist linear constant rank-distance  $k$  sets in  $H_n(\mathbb{F}_{q^2})$  of size  $q^k$  for all  $k$ . Note that if  $k < n$  is odd, these sets do not quite meet this new upper bound. This is due to the fact that we have no analogous subspace to  $S_\infty$  to add an extra element to the set  $\{S_A : A \in U\}$ .

For even rank  $k$ , we can always find constant rank-distance sets of size larger than the largest known linear constant rank-distance set, due to the following construction.

**Theorem 8.** *Suppose there exists a partial  $r$ -spread of  $\mathbb{F}_{q^2}^n$  of size  $N$ . Then there exists a constant rank-distance  $k = 2r$  set in  $H_n(\mathbb{F}_{q^2})$  of size  $N$ .*

*Proof.* Let  $k = 2r$ . Let  $D$  be a partial  $r$ -spread in  $\mathbb{F}_{q^2}^n$  of size  $N$ . To each  $S \in D$ , choose some matrix  $X_S \in M_{n \times r}$  whose column span is equal to  $S$  (for example, by choosing a basis for  $S$  and forming a matrix with these vectors as its columns). Next define

$$A_S = X_S \overline{X_S}^T \in H_n(\mathbb{F}_{q^2}).$$

Finally define  $U = \{A_S : S \in D\}$ . We claim that  $\text{rank}(A_S - A_T) = k$  for all  $S, T \in D$ ,  $S \neq T$ . For let  $A_S, A_T \in U$ . Then

$$\begin{aligned} A_S - A_T &= X_S \overline{X_S}^T - X_T \overline{X_T}^T \\ &= [X_S \quad X_T] \begin{bmatrix} I_r & 0_r \\ 0_r & 0_r \end{bmatrix} \begin{bmatrix} \overline{X_S}^T \\ \overline{X_T}^T \end{bmatrix} - [X_S \quad X_T] \begin{bmatrix} 0_r & 0_r \\ 0_r & I_r \end{bmatrix} \begin{bmatrix} \overline{X_S}^T \\ \overline{X_T}^T \end{bmatrix} \\ &= [X_S \quad X_T] \begin{bmatrix} I_r & 0_r \\ 0_r & -I_r \end{bmatrix} \begin{bmatrix} \overline{X_S}^T \\ \overline{X_T}^T \end{bmatrix} \end{aligned}$$

But  $S$  and  $T$  intersect trivially, and hence the matrix  $[X_S \quad X_T]$  has rank  $2r = k$ .

But  $\begin{bmatrix} I_r & 0_r \\ 0_r & -I_r \end{bmatrix}$  also has rank  $k$ , and hence  $A_S - A_T$  has rank  $k$ , as claimed. As shown in the introduction, this implies the existence of a constant rank-distance  $k$  set in  $H_n(\mathbb{F}_{q^2})$  of order  $|U| = N$ , proving the result.  $\square$

**Corollary 2.** *Suppose there exists a partial  $r$ -spread of  $\mathbb{F}_{q^2}^n$  of size  $N$ . Then there exists a constant dimension  $n$ , constant distance  $2k$  code in  $H(2n-1, q^2)$  of size  $N$ .*

**Remark 1.** Suppose  $q = k = 2$ ,  $n = 3$ . There exists a 1-spread of  $\mathbb{F}_{q^2}^3$  of size 21 (consisting of all 1-dimensional subspaces of  $\mathbb{F}_{q^2}^3$ ). A computer calculation using the computer algebra package MAGMA gave that the spectrum of sizes of maximal constant rank-distance 2 sets in  $H_3(\mathbb{F}_{2^2})$  is  $\{8, 10, 11, 12, 13, 14, 16, 17, 21\}$ . Hence the construction from Theorem 8 is maximal in this case. It is not clear whether this construction leads to maximal constant rank-distance sets in general.

**Remark 2.** Note that if  $n = k = 2r$ , there exists a spread consisting of all  $r$ -dimensional subspaces of  $\mathbb{F}_{q^2}^n$ , which has size  $\frac{q^{2n}-1}{q^{2r}-1} = q^n + 1$ . Hence this construction

gives a partial spread set in  $H_n(\mathbb{F}_{q^2})$  of size  $q^n + 1$ , and therefore a partial spread of size  $q^n + 2$  in  $H(2n - 1, q^2)$ , which is larger than the largest possible linear partial spread set.

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